

The effect of planetary migration on the corotation resonance

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ABSTRACT

The migration of a planet through a gaseous disc causes the locations of their resonant interactions to drift and can alter the torques exerted between the planet and the disc. We analyse the time-dependent dynamics of a non-coorbital corotation resonance under these circumstances. The ratio of the resonant torque in a steady state to the value given by Goldreich & Tremaine (1979) depends essentially on two dimensionless quantities: a dimensionless turbulent diffusion time-scale and a dimensionless radial drift speed. The dimensionless diffusion time-scale is a characteristic ratio of the time-scale of turbulent viscous diffusion across the librating region of the resonance to the time-scale of libration; in the absence of migration, this parameter alone determines the degree of saturation of the resonance. The dimensionless radial drift speed is the characteristic ratio of the drift speed of the resonance to the radial velocity in the librating region; this parameter determines the shape of the streamlines. When the drift speed is comparable to the libration speed and the viscosity is small, the torque can become much larger than the unsaturated value in the absence of migration, but is still proportional to the large-scale vortensity gradient in the disc. Fluid that is trapped in the resonance and drifts with it acquires a vortensity anomaly relative to its surroundings. If the anomaly is limited by viscous diffusion in a steady state, the resulting torque is inversely proportional to the viscosity, although a long time may be required to achieve this state. A further, viscosity-independent, contribution to the torque comes from fluid that streams through the resonant region. In other cases, torque oscillations occur before the steady value is achieved. We discuss the significance of these results for the evolution of eccentricity in protoplanetary systems. We also describe the possible application of these findings to the coorbital region and the concept of runaway (or type III) migration.

Key words: accretion, accretion discs — galaxies: kinematics and dynamics — hydrodynamics — planets and satellites: general

1 INTRODUCTION

Corotation resonances play an essential role in the gravitational interaction between a planet and the disc in which it forms. The perturbing potential associated with a planet (or other satellite) orbiting in a disc can be decomposed into a series of Fourier components with different azimuthal wavenumbers m and angular frequencies ω (Goldreich & Tremaine 1980). A fluid element in the disc at radius r , where the angular velocity is $\Omega(r)$, experiences a perturbing force with Doppler-shifted frequency $\hat{\omega} = \omega - m\Omega$. Resonant responses to horizontal forcing occur at radial locations where $\hat{\omega} = 0$ (the corotation resonance) and $\hat{\omega} = \pm\kappa$ (Lindblad resonances), where $\kappa(r)$ is the epicyclic

frequency in the disc. Under certain assumptions, including a linear approximation, simple formulae can be obtained for the localized resonant torques exerted at these locations (Goldreich & Tremaine 1979, hereafter GT79). The associated exchanges of angular momentum and energy cause the orbital parameters of the planet, in particular its semimajor axis and eccentricity, to evolve, and also play a role in shaping the mass distribution in the disc. These processes are of fundamental importance in determining the properties of planetary systems.

In a linear analysis the torque exerted at a corotation resonance is found to be proportional to the local radial gradient of vortensity (i.e. vertical vorticity divided by surface

density). However, the perturbed flow in the corotation region has islands of libration within which fluid is trapped and corotates, on average, with the potential. The mixing within these islands tends to erase the gradient of vortensity and reduce the corotation torque to zero (Goldreich & Tremaine 1981; Lubow 1990; Ward 1991, 1992). The resonance is then said to be completely saturated. Since viscous diffusion tends to reestablish the vortensity gradient, the level of saturation achieved in a steady state depends on the strength of the forcing and the viscosity. This behaviour is in contrast to that of Lindblad resonances, where the torque is insensitive to nonlinearity (Yuan & Cassen 1994) and can be reduced significantly only by removing mass from the resonant region.

In the case of a planet with a circular orbit, the angular pattern speed $\Omega_p = \omega/m$ of each non-axisymmetric potential component is equal to the angular velocity of the planet, and the corotation resonances are all *coorbital*, occurring at (very nearly) the same radial location as the planet's orbit. The coorbital corotation resonance is difficult to treat analytically because of the simultaneous presence of resonant components of all azimuthal wavenumbers and the high degree of nonlinearity. It is also in this region that three-dimensional effects are most important. The presence of a gap or partial clearing around the planet's orbit, together with shocks and accretion streams, complicates the problem further. An important study by Balmforth & Korycansky (2001) presented an asymptotic reduction of the coorbital region, allowing for a modest degree of nonlinearity, to a more tractable problem requiring the numerical solution of a partial differential equation. The time-evolution of the solution shows the formation of transient vortices followed by a mixing of vortensity across the corotation region, resulting in a saturation of the corotation torque.

Some recent direct numerical simulations of the interaction between a mobile planet and a disc have drawn attention to the importance of the coorbital region. Masset & Papaloizou (2003) found that very fast migration of a Saturn-mass protoplanet could occur in a disc somewhat more massive than the minimum-mass solar nebula. They presented a simplified analytical model allowing them to estimate the torque exerted by gas that crosses the planet's orbit as the planet migrates radially through the disc. Since this torque is proportional to the migration rate, it modifies (and typically reduces) the effective inertial mass of the planet, allowing it to migrate faster in response to non-coorbital torques. In addition, Masset & Papaloizou (2003) suggested that, if the effective inertial mass became negative and a time delay also occurred in its determination, the radial motion could be unstable and give rise to very fast (and generally inward) 'runaway' migration. Artymowicz (2004) presented a preliminary model for the torque due to the asymmetry in the librating region owing to planet migration, based on the behaviour of particle orbits. These effects have been described as 'type-III' migration to differentiate this regime from the previously identified regimes of migration of embedded planets (type I) and gap-opening planets (type II) (Ward 1997). Recently the numerical results of Masset & Papaloizou (2003) have been challenged by D'Angelo, Bate & Lubow (2005), who found that the effect is strongly or completely suppressed when the region

close to the planet is better resolved. The reality of runaway migration is therefore in some doubt.

In the case of a planet with an eccentric orbit, potential components are present at first order in the eccentricity that have angular pattern speeds different from the mean motion of the planet. The principal effect of the associated corotation and Lindblad torques is to cause the eccentricity to evolve. Provided that a deep gap is cleared around the planet, so that the coorbital eccentric Lindblad resonances are ineffective, there is a fine balance between the growth of eccentricity through eccentric Lindblad resonances and its decay through eccentric corotation resonances, which are non-coorbital (Goldreich & Tremaine 1980; Ward 1988), if many such resonances are able to compete with each other. Even a partial saturation of the corotation torques therefore promotes the growth of the planet's eccentricity. It should be noted, however, that whether the eccentricity grows or decays depends on a number of additional factors. Goldreich & Sari (2003) discussed some of the relevant issues but further work is required to treat the development of eccentricity in the disc and the conservative secular exchange of eccentricity between the planet and the disc.

We recently analysed the saturation of the non-coorbital corotation resonance in a gaseous disc (Ogilvie & Lubow 2003, hereafter Paper I). Unlike the calculations of Masset (2001) and Masset & Papaloizou (2003), which, owing to the extreme complexity of the coorbital region, necessarily have the nature of a toy model, our calculation is based on an asymptotically exact reduction of the problem making minimal assumptions. We showed that the steady corotation torque for any non-coorbital resonance is reduced below the value specified by GT79 by a factor $t_c(p)$ depending on a single dimensionless parameter p , which measures the strength of the forcing relative to the effects of viscosity in the disc. We computed the function $t_c(p)$ numerically and derived analytical approximations for small and large p . Our result was used by Goldreich & Sari (2003) in their theory of the growth of the eccentricity of a protoplanet through a finite-amplitude instability, and our findings have recently been corroborated by Masset & Ogilvie (2004) using localized numerical simulations.

The principal aim of this paper is to extend the analysis of Paper I to include the effect of planetary migration. We also examine the time-dependent approach to a steady state and emphasize the interpretation of the solutions in real space rather than Fourier space. Finally, we attempt to relate our findings in a preliminary way to the coorbital region and the issue of runaway migration.

2 INCLUSION OF TIME-DEPENDENCE

In Paper I we carried out a systematic asymptotic analysis of the corotation region of a three-dimensional, barotropic, viscous, non-self-gravitating disc subject to a uniformly rotating external potential perturbation. To calculate the steady torque exerted on the disc, we sought a solution that is steady in the corotating frame of reference. We made use of the small parameter ϵ , which is a characteristic value of the angular semithickness H/r of the disc. In units such that the corotation radius and the corresponding angular velocity are of order unity, we adopted natural scalings such that

the width and height of the corotation region are $O(\epsilon)$. We introduced scaled radial and vertical coordinates, ξ and ζ , to resolve the inner structure of this region.

It is a simple matter to restore time-dependence to the original problem. The characteristic time-scale for establishing the steady solution is $O(\epsilon^{-1})$, this being typical of both the libration time-scale and the viscous diffusion time-scale across the corotation region under our scaling assumptions. We therefore allow the solution to depend on a ‘slow’ time variable $\tau = \epsilon t$, and the effect is to replace each instance of the operator

$$\left(u'_0 \frac{\partial}{\partial \xi} + \Omega_1 \xi \frac{\partial}{\partial \phi} \right), \quad (1)$$

as in equation (17) of Paper I, with

$$\left(\frac{\partial}{\partial \tau} + u'_0 \frac{\partial}{\partial \xi} + \Omega_1 \xi \frac{\partial}{\partial \phi} \right). \quad (2)$$

A further adaptation allows for a ‘very slow’ time-dependence of the external potential. This is natural if we consider a Fourier component of the tidal potential of a planet that migrates through the disc on a time-scale that is long compared to the orbital time-scale and also compared to the characteristic time-scales of libration and viscous diffusion across the corotation region. In this case both the radial structure and the angular pattern speed of the potential evolve very slowly in time. We allow for this formally by introducing a very slow time variable $T = \epsilon^2 t$ and writing the potential perturbation in the midplane $z = 0$ as

$$\Phi' = \Phi'(r, \varphi, T), \quad (3)$$

where

$$\varphi = \phi - \epsilon^{-2} \int \Omega_p(T) dT, \quad (4)$$

ϕ being the usual azimuthal angle in an inertial frame of reference. The pattern speed of the potential is then $\Omega_p(T)$, a very slowly varying angular frequency of order unity, and the corotation radius is $r_c(T)$, defined by the condition $\Omega(r_c(T)) = \Omega_p(T)$.

We transform from spatial coordinates (r, ϕ, z) to (ξ, φ, ζ) , with

$$\xi = \frac{r - r_c(T)}{\epsilon}, \quad \zeta = \frac{z}{\epsilon}. \quad (5)$$

Using the chain rule we find

$$\frac{\partial}{\partial t} \mapsto \epsilon \frac{\partial}{\partial \tau} + \epsilon^2 \frac{\partial}{\partial T} - \epsilon \frac{dr_c}{dT} \frac{\partial}{\partial \xi} - \Omega_p \frac{\partial}{\partial \varphi}, \quad (6)$$

$$\frac{\partial}{\partial r} \mapsto \epsilon^{-1} \frac{\partial}{\partial \xi}, \quad (7)$$

$$\frac{\partial}{\partial \phi} \mapsto \frac{\partial}{\partial \varphi}, \quad (8)$$

$$\frac{\partial}{\partial z} \mapsto \epsilon^{-1} \frac{\partial}{\partial \zeta}. \quad (9)$$

We proceed to solve the fluid dynamical equations as in Paper I by expanding the solution in powers of ϵ . Whereas previously we looked for a strictly steady solution in the corotating frame, now the solution depends on the slow and very slow time variables τ and T . The time-derivative associated with the dependence on T is small, $O(\epsilon^2)$, and does not affect the equations to the order that we considered. On

the other hand, there is a new ‘advective’ time-derivative $-(dr_c/dt)(\partial/\partial \xi)$ at $O(\epsilon)$ that does affect the analysis, along with the derivative with respect to τ . The net effect is that the operator (1) is replaced with

$$\left[\frac{\partial}{\partial \tau} + \left(u'_0 - \frac{dr_c}{dT} \right) \frac{\partial}{\partial \xi} + \Omega_1 \xi \frac{\partial}{\partial \varphi} \right]. \quad (10)$$

By analyzing the problem in a frame of reference that moves with the resonance (hereafter referred to as the comoving frame), we transform the planetary migration into a uniform radial drift of the gas through the resonant region (cf. Masset & Papaloizou 2003).

When we remove the ϵ -scalings and present the reduced equation for the enthalpy perturbation $h'(x, \varphi, t)$ in dimensional form, we obtain

$$\begin{aligned} & \left[\frac{\partial}{\partial t} + \left(u' - \frac{dr_c}{dt} \right) \frac{\partial}{\partial x} + \frac{d\Omega}{dr} x \frac{\partial}{\partial \varphi} \right] \left(\frac{\kappa^2}{c^2} - \frac{\partial^2}{\partial x^2} \right) h' \\ & + \nu \left(\frac{\partial^2}{\partial x^2} + 2r\Omega \frac{d\Omega}{dr} \frac{D_\mu}{c^2} \right) \frac{\partial^2 h'}{\partial x^2} \\ & = \frac{2\Omega}{r} \left(\frac{\partial \Phi'}{\partial \varphi} \right) \frac{d}{dr} \ln \left(\frac{\Sigma}{B} \right), \end{aligned} \quad (11)$$

where we recall that $x = r - r_c$ is the radial distance from corotation,

$$u' = -\frac{1}{2rB} \frac{\partial \Phi'}{\partial \varphi} \quad (12)$$

is the leading-order radial velocity perturbation induced by the tidal potential Φ' , $2B = (1/r)(d/dr)(r^2\Omega)$ is the vertical vorticity of the unperturbed disc, c is a certain vertical average of the sound speed, ν is the mean kinematic viscosity, Σ is the surface density and $D_\mu = \partial \ln(\nu\Sigma)/\partial \ln \Sigma$. In equation (11) all coefficients are to be regarded as independent of x and evaluated at $r = r_c$, except in the one place where x appears explicitly. The quantities Φ' and u' are regarded as functions of φ only. The time-dependence of the coefficients is neglected, on the basis that the time-scale for establishing the solution is short (measured by τ) compared to that on which the coefficients vary significantly (measured by T).

As in Paper I, we consider a single potential component of the form

$$\Phi' = \Psi \cos(m\varphi), \quad (13)$$

and the solution for h' may be assumed to have the same periodicity in φ . The total tidal torque exerted in the corotation region can be written as

$$T_c(t) = -\frac{r\Sigma}{c^2} \int_0^{2\pi} \int_{-\infty}^{\infty} h'(x, \varphi, t) \frac{\partial \Phi'}{\partial \varphi} dx d\varphi. \quad (14)$$

The boundary condition as $|x| \rightarrow \infty$ is that $\partial h'/\partial x \rightarrow 0$. This ensures that the disturbance is localized in the corotation region in the sense that the vortensity gradient at large x is just that of the unperturbed disc. Generally, h' does not tend to zero as $|x| \rightarrow \infty$ (although it is bounded) because the corotation torque must be balanced in a steady state by an adjustment of the viscous stress (and therefore the surface density) across the corotation region.

3 REDUCTION TO A DIMENSIONLESS FORM

As in Paper I, we rewrite equation (11) in a dimensionless form by means of the transformations

$$t = \tilde{t} \frac{\kappa}{mc} \left(-\frac{dr}{d\Omega} \right), \quad x = \tilde{x} \frac{c}{\kappa}, \quad \varphi = \frac{\theta}{m}, \quad (15)$$

$$h' = f(\tilde{x}, \theta, \tilde{t}) \frac{c^3}{\kappa} \frac{d}{dr} \ln \left(\frac{\Sigma}{B} \right). \quad (16)$$

We then obtain

$$\left[-\frac{\partial}{\partial \tilde{t}} + (d - a \sin \theta) \frac{\partial}{\partial \tilde{x}} + \tilde{x} \frac{\partial}{\partial \theta} \right] \left(1 - \frac{\partial^2}{\partial \tilde{x}^2} \right) f - \tilde{\nu} \left(\frac{\partial^2}{\partial \tilde{x}^2} - b \right) \frac{\partial^2 f}{\partial \tilde{x}^2} = a \sin \theta, \quad (17)$$

where

$$a = 2 \left(-\frac{d \ln r}{d \ln \Omega} \right) \frac{\Psi}{c^2}, \quad (18)$$

$$b = -2r\Omega \frac{d\Omega}{dr} \frac{D_\mu}{\kappa^2}, \quad (19)$$

$$\tilde{\nu} = \frac{1}{m} \left(-\frac{dr}{d\Omega} \right) \frac{\kappa^3}{c^3} \nu, \quad (20)$$

$$d = \frac{1}{m} \left(-\frac{dr}{d\Omega} \right) \frac{\kappa^2}{c^2} \frac{dr_c}{dt}. \quad (21)$$

The ratio

$$v = \frac{d}{a} = \frac{2rB}{m\Psi} \frac{dr_c}{dt} \quad (22)$$

measures the relative effect of the new advective term associated with the drift of the corotation resonance.

The corotation torque can be written in the form

$$T_c = t_c T_{GT}, \quad (23)$$

where t_c is dimensionless, and

$$T_{GT} = \frac{m\pi^2 \Psi^2}{2(d\Omega/dr)} \frac{d}{dr} \left(\frac{\Sigma}{B} \right) \quad (24)$$

is the torque formula of GT79.

4 ANALYSIS IN REAL SPACE

4.1 Interpretation as a vortensity equation

In a two-dimensional, barotropic, inviscid flow, the vorticity equation takes the form

$$\frac{D}{Dt} \left(\frac{\omega_z}{\Sigma} \right) = 0, \quad (25)$$

where $\omega_z = \mathbf{e}_z \cdot (\nabla \times \mathbf{u})$ is the (vertical component of) vorticity. The quantity ω_z/Σ is sometimes referred to as the potential vorticity or (among accretion disc theorists) the vortensity. It is not generally conserved in a three-dimensional flow, nor is it conserved in the presence of viscosity.

We consider the related quantity $Q = \ln(\Sigma/\omega_z)$. In the unperturbed disc this is equal to $\bar{Q} = \ln(\Sigma/B) - \ln 2$ and has a purely radial gradient $d\bar{Q}/dr = (d/dr) \ln(\Sigma/B)$. This gradient appears in equation (11) and can be regarded as constant across the corotation region. In the perturbed disc, under our scaling assumptions,

$$Q = \bar{Q} + Q' = \bar{Q} + \frac{h'}{c^2} - \frac{1}{\kappa^2} \frac{\partial^2 h'}{\partial x^2}. \quad (26)$$

Equation (11) cannot generally be written as a closed equation for Q because of the viscous terms. However, a simplification occurs when the dimensionless parameter

$$b = -2r\Omega \frac{d\Omega}{dr} \frac{D_\mu}{\kappa^2}, \quad (27)$$

equal to $3D_\mu$ in a Keplerian disc, is equal to unity. As in Paper I, we adopt this convenient assumption, anticipating that our results will not depend sensitively on it. When $b = 1$, equation (11) can be divided through by κ^2 and interpreted in the form

$$\frac{\partial Q'}{\partial t} + \mathbf{u} \cdot \nabla Q' - \nu \frac{\partial^2 Q'}{\partial x^2} = -u' \frac{d\bar{Q}}{dr}, \quad (28)$$

where

$$\mathbf{u} = \left(u' - \frac{dr_c}{dt} \right) \mathbf{e}_r + r \frac{d\Omega}{dr} x \mathbf{e}_\varphi \quad (29)$$

is the velocity field in the comoving frame correct to a certain level of approximation.¹ Equation (28) is equivalent to

$$\frac{DQ}{Dt} = \nu \nabla^2 Q, \quad (30)$$

provided that the diffusive ∇^2 operator is considered to act only on the more rapid variation of the perturbation Q' in the radial direction. It should be noted that the quantity $D\bar{Q}/Dt$ does not contain any contribution from the drift; it is a Galilean-invariant quantity and is most easily evaluated in the non-moving frame. Evidently, equation (30) derives from a version of potential vorticity conservation. Three-dimensional effects are absent for the flows under consideration, which are quasi-two-dimensional because the motion is predominantly horizontal and maintains a quasi-hydrostatic balance in the vertical direction. Only when $b = 1$ do the viscous terms have a simple closed form in terms of Q .

4.2 Streamlines of the dominant motion

It is of interest to plot the streamlines of the velocity field (29), which are contour lines of the streamfunction

$$\chi(x, \varphi) = -\frac{\Psi}{2B} \cos(m\varphi) - r \frac{dr_c}{dt} \varphi - \frac{r}{2} \frac{d\Omega}{dr} x^2. \quad (31)$$

These are not the exact streamlines of the fluid but represent the dominant motion $\mathbf{u} = \nabla \chi \times \mathbf{e}_z$ in the comoving frame. In the rescaled variables of Section 3, the streamfunction is proportional to

$$-a \cos \theta - d\theta + \frac{1}{2} \tilde{x}^2, \quad (32)$$

and the shape of the streamlines depends only on the dimensionless parameter

$$v = \frac{d}{a} = \frac{2rB}{m\Psi} \frac{dr_c}{dt}, \quad (33)$$

introduced previously, which measures the drift speed relative to the characteristic libration speed. The streamlines

¹ Since the solution varies more rapidly in the radial direction than in the azimuthal direction, u_r is in fact correct to $O(\epsilon^2)$ while u_φ is correct to $O(\epsilon)$.

are plotted in Fig. 1 for the cases $v = 0, 0.5, 1$ and 2 (those for negative values of v can be obtained by a reflection in the x -axis). Stagnation points occur where $x = 0$ and $\sin \theta = v$, which yields two solutions in $0 \leq \theta < 2\pi$ if $|v| < 1$ and none if $|v| > 1$. In the absence of drift, the streamlines circulate both interior and exterior to the resonant radius, but form symmetric librating islands centred on the resonance. The drift breaks the symmetry of the flow with respect to the x -axis. If the drift is not too fast ($|v| < 1$) the librating islands remain but they become asymmetrical and are diminished in size; this opening allows gas to flow through from the outer disc to the inner disc (if $v > 0$). A fast drift ($|v| > 1$) destroys the librating islands and allows a free streaming across the corotation region. For a given strength of potential there is therefore a critical migration rate that changes the topology of the flow and inhibits the libration normally associated with the corotation resonance.

4.3 Behaviour of the vortensity perturbation

Now equation (28) can be interpreted as an advection–diffusion equation with a steady source term. The quantity Q' that is being advected and diffused is (minus) the fractional vortensity perturbation, and the source term derives from the velocity perturbation u' induced by the forcing. In the presence of viscosity a steady solution will be approached in which the advective and diffusive terms balance the source term, i.e.

$$\mathbf{u} \cdot \nabla Q' - \nu \frac{\partial^2 Q'}{\partial x^2} = -u' \frac{d\bar{Q}}{dr}. \quad (34)$$

In Appendix A we analyse the solution of an equation of this type in the limit of small viscosity. The conclusion is that the perturbation builds up on the closed streamlines of the librating region to a value, proportional to ν^{-1} , at which it is equilibrated by outward diffusion to the region of open streamlines. We apply equation (A14) of Appendix A by setting the source term equal to $-\mathbf{u} \cdot \nabla \bar{Q} - (dr_c/dt) d\bar{Q}/dr$ and noting that the first contribution integrates to zero. Also $\nabla^2 \chi$ is replaced by the constant value $\partial^2 \chi / \partial x^2$ because the radial derivatives are asymptotically dominant. Therefore

$$Q' \approx \frac{1}{\nu r} \left(-\frac{dr}{d\Omega} \right) \frac{dr_c}{dt} \frac{d\bar{Q}}{dr} (\chi - \chi_s) \quad (35)$$

in the librating region in a steady state, where χ_s is the streamfunction of the separatrix. In the region of open streamlines outside,

$$Q' \approx \int_{-\infty}^{\infty} \left(-u' \frac{d\bar{Q}}{dr} \right) d\lambda, \quad (36)$$

where the integral is taken along the streamline from far upstream, and λ is the travel time along the streamline.

4.4 Estimation of the torque in the low-viscosity limit

The corotation torque (14) can also be expressed in terms of Q' , noting that the last term in equation (26) integrates to zero:

$$T_c = -r\Sigma \int_0^{2\pi} \int_{-\infty}^{\infty} Q' \frac{\partial \Phi'}{\partial \varphi} dx d\varphi$$

$$= -\frac{2rB\Sigma}{d\bar{Q}/dr} \int Q' \left(-u' \frac{d\bar{Q}}{dr} \right) dA. \quad (37)$$

In the low-viscosity limit, the torque in a steady state is then (again using the results of Appendix A)

$$T_c \approx -\frac{2rB\Sigma}{\nu} \left(\frac{dr_c}{dt} \right)^2 \frac{d\bar{Q}}{dr} \int x^2 dA - \frac{m^2 \Psi^2 \Sigma}{2rB} \frac{d\bar{Q}}{dr} \int \frac{1}{2} \left[\int_{-\infty}^{\infty} \sin \theta d\lambda \right]^2 d\chi, \quad (38)$$

where the first integral is over the librating region and the second is over the region of open streamlines.

The separatrix bounding the librating region is the streamline passing through the saddle point at $x = 0$, $\theta = \theta_s = \pi - \arcsin v$. The equation of the separatrix can be written in dimensionless form as

$$x = \pm \left[\frac{\Psi}{rB} \left(-\frac{dr}{d\Omega} \right) \right]^{1/2} X(\theta), \quad (39)$$

where $X(\theta)$ is defined by

$$X^2 = \cos \theta - \cos \theta_s + v(\theta - \theta_s) \quad (40)$$

and is positive in the interval $\theta_1 < \theta < \theta_s$, where θ_1 is the other value of θ at which the separatrix crosses the x -axis. After some rearrangement, and remembering that there are m copies of the librating region, we find the ratio of the torque to the GT79 value in the low-viscosity limit to be

$$t_c \approx \frac{2\sqrt{2}}{\pi^2} \frac{d^2}{a^{1/2}\tilde{\nu}} \int_{\theta_1}^{\theta_s} \frac{2}{3} [X(\theta)]^3 d\theta + \frac{1}{\pi^2} \int \frac{1}{2} \left[\int_{-\infty}^{\infty} \sin \theta d\tilde{\lambda} \right]^2 d\tilde{\chi}, \quad (41)$$

where the second integral is over a single copy of the streaming region, and the dimensionless streamfunction and travel time are defined by

$$\chi = \tilde{\chi} \frac{rc^2}{\kappa^2} \left(-\frac{d\Omega}{dr} \right), \quad \lambda = \tilde{\lambda} \frac{\kappa}{mc} \left(-\frac{dr}{d\Omega} \right). \quad (42)$$

4.5 Interpretation of the torques

The physical meaning of this analysis is that, if the disc has a large-scale vortensity gradient, fluid that is trapped in the corotation resonance and forced to drift with it attempts to preserve its original vortensity. A steady state is achieved when the vortensity anomaly is equilibrated by viscous diffusion to the fluid outside the librating region. If the viscosity is small, the torque comes mainly from the librating region (the first term in equation 38 or equation 41) but there is a further contribution, differing from the GT79 torque only by a factor of order unity, from the fluid that streams through the resonance (the second term in the same equations).

The first integral in equation (41) is a dimensionless quantity of order unity that decreases monotonically from $32\sqrt{2}/9 \approx 5.028$ to 0 as $|v|$ increases from 0 to 1 and the librating region shrinks (cf. Fig. 1). Therefore the first contribution to t_c scales as $d^2 a^{-1/2} \tilde{\nu}^{-1}$ for small v . In dimensional terms this contribution to the torque scales as

$$\Psi^{3/2} \left(-\frac{d \ln \Omega}{d \ln r} \right)^{-3/2} \frac{r^2 \kappa}{\Omega^2 \nu} \left(\frac{dr_c}{dt} \right)^2 \frac{d}{dr} \left(\frac{\Sigma}{B} \right). \quad (43)$$

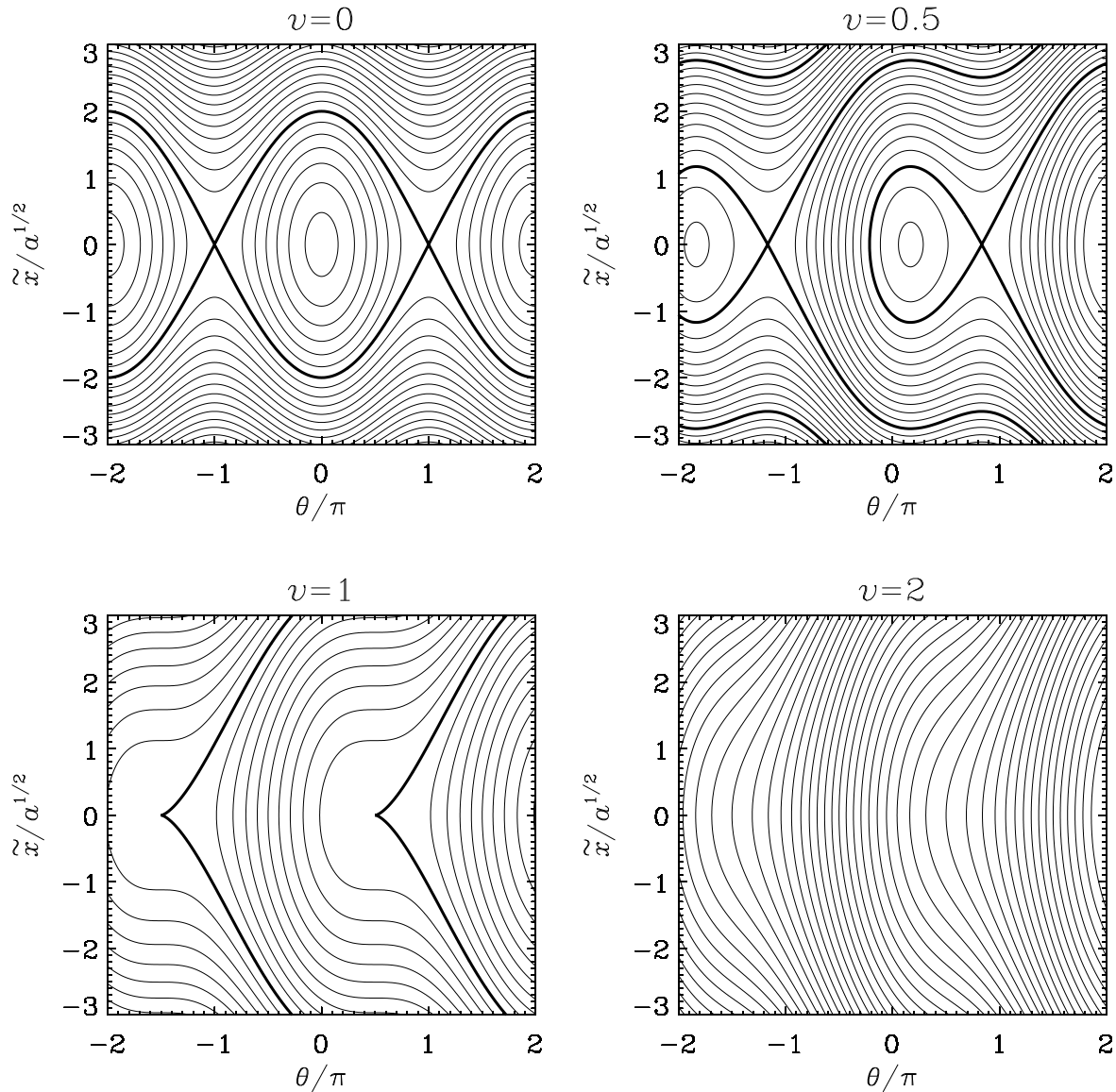


Figure 1. Streamlines of the dominant motion in the comoving frame, for various values of the drift-to-libration ratio $v = d/a$. The separatrices are marked as bold lines. A librating region of closed streamlines exists for $|v| < 1$. The variable θ is shown over two periods for greater clarity; the actual number of periodic copies is m . The increment of streamfunction between neighbouring contours is twice as large in the lower panels as in the upper panels, implying that the velocities are typically larger.

A simple order-of-magnitude argument for this result can be given as follows. In the absence of turbulent diffusion, the vortensity in the librating region remains constant as the planet migrates. However, diffusion limits the contrast in Q between the librating region and the background to $\Delta Q \sim -(dr_c/dt)t_\nu(d\bar{Q}/dr)$, where $t_\nu = \delta^2/\nu$ is the viscous time-scale across the librating region of radial width δ for which $\delta^2 \sim \Psi/[rB(-d\Omega/dr)]$. Then the torque in equation (43) can be understood as $\sim -(\Delta Q)(\Sigma r\delta)(rB)(dr_c/dt)$. This is only a fraction of order $-\Delta Q$ of the rate of change of orbital angular momentum of the trapped region. The reason that only a fraction is required is that the region occupied by the

moving trapped fluid is replenished by fluid of the ambient vortensity.²

The angular momentum taken up by the vortensity anomaly is somewhat analogous to the kinetic energy developed by the entropy anomaly in stellar convection. Both

² In the theory of vortex dynamics for a two-dimensional ideal incompressible fluid (e.g. Lamb 1932), a conserved quantity playing the role of angular momentum is the angular impulse $-\frac{1}{2}\Sigma \int \omega_z r^2 dA$. The torque required to move a small vortex patch of mass m and strength ω_z and mass m radially at speed \dot{r} is $-m\omega_z r\dot{r}$. This is a fraction $-2\omega_z/\Omega$ of the torque required to move radially a particle of mass m in a circular Keplerian orbit of angular velocity Ω .

situations involve the interchange of perturbed and ambient material. An equation that is analogous to equation (43) holds in the case of convection (see Schwarzschild 1958). In that case, the kinetic energy is proportional to the unperturbed entropy gradient (analogue of the vortensity gradient), gravity (analogue of the angular momentum gradient) and the square of the mixing length (analogue of the drift speed).

Evidently this analysis may break down if the viscosity is too small, for then the anomaly ΔQ will not be small and further nonlinearity may intervene. In addition, if the viscosity is very small, this steady state may take so long to be achieved that the torque needs to be considered in a time-dependent sense. However, a possible way to rationalize the large value of t_c obtained in the low-viscosity limit is as follows. Suppose that the resonance drifts outwards in a disc in which Σ is a decreasing function of r . By the time a steady state is reached, the ambient surface density, and therefore T_{GT} , may be very small. However, the torque may be significant because it derives from the trapped fluid. In this case t_c will be very large, although it will not be described accurately by the approximations we have adopted.

The second torque integral in equation (41), which derives from the open streamlines, is also a dimensionless quantity of order unity. In the limit of fast streaming ($|v| \gg 1$) the dimensionless streamfunction and travel time may be approximated as $\tilde{\chi} \approx -d\theta + \frac{1}{2}\tilde{x}^2$ and $\tilde{\lambda} = -\tilde{x}/d$. The second integral in equation (41) then reduces to a standard Fresnel integral with the result that $t_c \rightarrow 1$. The unsaturated GT79 torque is therefore recovered in the limit of fast migration.

5 ANALYSIS IN FOURIER SPACE

As in Paper I, we apply a Fourier analysis in \tilde{x} and θ , writing

$$f(\tilde{x}, \theta, \tilde{t}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\frac{i\pi a}{1+k^2} \right) \sum_{n=-\infty}^{\infty} G_n(k) e^{ik\tilde{x} + in\theta} dk. \quad (44)$$

Equation (36) of Paper I then becomes

$$\left[\frac{\partial}{\partial \tilde{t}} - idk + n \frac{\partial}{\partial k} + \tilde{\nu} k^2 \left(\frac{b+k^2}{1+k^2} \right) \right] G_n - \frac{ka}{2} (G_{n+1} - G_{n-1}) = (\delta_{n,1} - \delta_{n,-1}) \delta(k). \quad (45)$$

The total tidal torque exerted in the corotation region (equation 38 of Paper I) is, in dimensionless form,

$$t_c = G_1(0) - G_{-1}(0). \quad (46)$$

Under the simplifying assumption $b = 1$ (Paper I) it is natural to rescale the wavenumber and time variable to

$$\tilde{k} = \tilde{\nu}^{1/3} k, \quad t_* = \tilde{\nu}^{1/3} \tilde{t}, \quad (47)$$

so that

$$\left(\frac{\partial}{\partial t_*} - 2ivp\tilde{k} + n \frac{\partial}{\partial \tilde{k}} + \tilde{k}^2 \right) G_n - p\tilde{k} (G_{n+1} - G_{n-1}) = (\delta_{n,1} - \delta_{n,-1}) \delta(\tilde{k}), \quad (48)$$

where

$$p = \frac{1}{2} a \tilde{\nu}^{-2/3}$$

$$= \left(\frac{\Psi}{-\kappa^2 d \ln \Omega / d \ln r} \right) \left(\frac{-m d\Omega / dr}{\nu} \right)^{2/3}. \quad (49)$$

The dimensionless parameter p can be understood as the ratio of the characteristic time-scale of viscous diffusion across the librating region of the resonance to the characteristic time-scale of libration (raised to the power 2/3). In Paper I we found that p alone determines the degree of saturation of the resonance in the absence of migration.

Linear theory applies when $p \ll 1$, although vp need not be small. Only modes $n = \pm 1$ are then significantly excited, and the solution in a steady state is

$$G_{\pm 1} = \pm H(\pm \tilde{k}) \exp \left[\mp \left(\frac{1}{3} \tilde{k}^3 - ivp \tilde{k}^2 \right) \right], \quad (50)$$

giving rise to a dimensionless torque $t_c = 1$, i.e. $T_c = T_{GT}$. The corotation torque is therefore unaffected by planetary migration when $p \ll 1$, although the form of the disturbance is changed into a damped wave propagating radially downstream of the resonance.

6 METHODS OF NUMERICAL SOLUTION

We solve equation (48) and evaluate the corotation torque by two independent methods. The first method is identical to that described in Section 3.10 of Paper I, in which we obtain steady solutions by setting the time-derivative to zero and solving the (truncated) system of ordinary differential equations by a shooting method.

The second method treats the equations as an initial-value problem, for which the initial condition corresponding to an unperturbed disc is $G_n(\tilde{k}, 0) = 0$. The operator $(\partial/\partial t_*) + n(\partial/\partial \tilde{k})$ in equation (48) can be regarded as a Lagrangian derivative following a shearing flow in the semi-discrete (n, \tilde{k}) Fourier space. We therefore discretize the problem with respect to \tilde{k} by introducing a shearing lattice of Fourier wavevectors,

$$\tilde{k}_{n,j}(t_*) = j \delta \tilde{k} + nt_*, \quad (51)$$

where j is an integer index and $\delta \tilde{k}$ is the lattice spacing. As the lattice shears, it periodically regains its form with a period of $\delta t_* = \delta \tilde{k}$. The solution G is represented by its values $G_{n,j}(t_*)$ at the moving lattice points, so that equation (48) becomes

$$\begin{aligned} & \left(\frac{d}{dt_*} - 2ivp\tilde{k}_{n,j} + \tilde{k}_{n,j}^2 \right) G_{n,j} \\ & - p\tilde{k}_{n,j} [G_{n+1,j-(t_*/\delta t_*)} - G_{n-1,j+(t_*/\delta t_*)}] \\ & = \delta_{n,1} \delta(t_* + j \delta t_*) - \delta_{n,-1} \delta(t_* - j \delta t_*), \end{aligned} \quad (52)$$

while the dimensionless torque is given by

$$t_c = G_{1, -(t_*/\delta t_*)} - G_{-1, (t_*/\delta t_*)}. \quad (53)$$

In this representation, each Fourier amplitude starts at zero at $t_* = 0$; the wavevector evolves according to the motion of the shearing lattice and the amplitude becomes non-zero either through the coupling terms (proportional to p) or through direct forcing (if $n = \pm 1$). Now $t_*/\delta t_*$ is not generally an integer, so some interpolation is required for the coupling terms; we use a simple linear interpolation between the bracketing integer values.

It is unnecessary to follow the wavevectors when they shear to large values of k because they become very strongly damped by viscosity. We apply a truncation such that $|j| \leq J$, $|n| \leq N$ and periodically remap the wavenumbers, which is equivalent to resetting t_* back to zero every time it reaches δt_* .

7 NUMERICAL RESULTS

The corotation torque in a steady state is shown in Fig. 2. For each value of the parameter p there is a certain degree of saturation of the torque at $v = 0$, as described in Paper I. As $|v|$ increases from zero the torque increases and can become significantly larger than the GT79 value (i.e. $t_c > 1$) if the viscosity is small enough (i.e. if p is large enough). This enhanced torque derives from the librating region where a viscously equilibrated vortensity anomaly is established. As $|v|$ is increased further the torque is reduced because the librating region shrinks and disappears at $|v| = 1$. Therefore the torque is maximized for $|v| \approx 1/2$. For $|v| > 1$ the torque slightly exceeds the GT79 value for any value of p . In this limit the torque derives from gas that streams through the resonance, and no saturation occurs.

The general form of the numerically determined steady torque agrees fairly well with the approximate relation (41), confirming our interpretation. It appears, however, that the convergence to the limiting form as the viscosity is reduced is rather slow.

The time-dependence of the corotation torque, starting from an unperturbed disc, is shown for some representative cases in Fig. 3. The case $v = 0$ is reminiscent of Balmforth & Korycansky (2001): the torque undergoes damped oscillations as the librating fluid stirs and mixes the vortensity, leading to saturation of the resonance. For small p , no oscillations occur and the degree of saturation is slight. For large p , many oscillations occur and the final steady torque is much reduced.

The case $v = 0.5$ illustrates what happens when a substantial asymmetrical librating region exists. The dynamics of that region no longer leads to a cancellation of the torque because of the systematic vortensity anomaly that builds up and is limited only by viscous diffusion. For large p , the time required to reach a steady state is considerable and the final torque is much enhanced.

Finally, the case $v = 2$ corresponds to a situation of fast migration in which no librating region occurs. The disturbance now takes the form of a kinematic wave downstream of the resonance. Care must be taken that the spatial domain is large enough that the wave is viscously attenuated before it reaches the boundary. For large p some oscillations in the torque occur but the result differs only slightly from the GT79 value. Good agreement is found between the results obtained with independent numerical methods.

8 APPLICATION TO ECCENTRIC RESONANCES

As discussed in Paper I and by Goldreich & Sari (2003), the modification of the corotation torque has consequences for the eccentricity evolution of young planets orbiting in

a protoplanetary disc. We consider the first-order eccentric corotation resonances associated with a planet of mass ratio $q = M_p/M_*$ executing an orbit of eccentricity e and semimajor axis a_p within a Keplerian disc. Inner and outer resonances occur at radii

$$r_c = \left(\frac{m}{m \pm 1} \right)^{2/3} a_p, \quad (54)$$

and the associated torques lead to eccentricity damping. The corresponding dimensionless parameters p and v for a migrating planet are

$$p = 0.7006 C_m^\pm m^{5/3} e q \alpha^{-2/3} \left(\frac{H}{r} \right)^{-4/3}, \quad (55)$$

$$v = \frac{1}{1.604 C_m^\pm m^2 e q} \left(\frac{dr_c/dt}{r\Omega} \right), \quad (56)$$

where the notation is as in Paper I (C_m^\pm being a correction factor that tends to unity for large m). Furthermore

$$vp = 0.6552 \left[\frac{\alpha}{m} \left(\frac{H}{r} \right) \right]^{1/3} \left(\frac{dr_c/dt}{u_\nu} \right), \quad (57)$$

where $u_\nu = (3/2)(\nu/r)$ is the characteristic magnitude of the radial velocity in a Keplerian accretion disc.

In type-II migration the factor $(dr_c/dt)/u_\nu$ is close to unity while the factor in square brackets is very small. Either v or p must be small, so the large enhancement of the torque seen in Fig. 2 does not occur. It is possible to have $|v| > 1$ if e is sufficiently small. This situation occurs because the librating region is narrow and the libration speed small, so the dimensionless drift speed is large. But then $p \ll 1$, since the dimensionless diffusion time-scale is short, and so $t_c \approx 1$ whether the planet is migrating or not. Therefore planetary migration in the type-II regime is unimportant for eccentricity evolution, at least as far as the eccentric corotation resonances are concerned.

In type-I migration the factor $(dr_c/dt)/u_\nu$ can be much larger than unity and vp need not be small. Because of the dependence of v on m it is likely that the large enhancement of the torque is restricted to a few values of m at most. The issue is less important here because in the type-I regime the eccentricity damping is dominated by coorbital Lindblad resonances.

9 TENTATIVE APPLICATION TO THE COORBITAL REGION

9.1 Approximate flow in the coorbital region

We now consider the possible application of these results to the more difficult problem of the coorbital region for the case of a planet of mass M_p in a circular orbit of radius $r_p(t)$ undergoing radial migration. The perturbing potential in this case, including the indirect term, is

$$\begin{aligned} \Phi' = & -GM_p(r^2 + r_p^2 - 2rr_p \cos \varphi)^{-1/2} \\ & + GM_p r_p^{-2} r \cos \varphi, \end{aligned} \quad (58)$$

and the corotation radius is $r_c = (1+q)^{-1/3} r_p$, where $q = M_p/M_*$ is the mass ratio. For $q \ll 1$, and in the coorbital region where $x = r - r_c$ satisfies $|x| \ll r_c$, the perturbing potential can be adequately approximated as

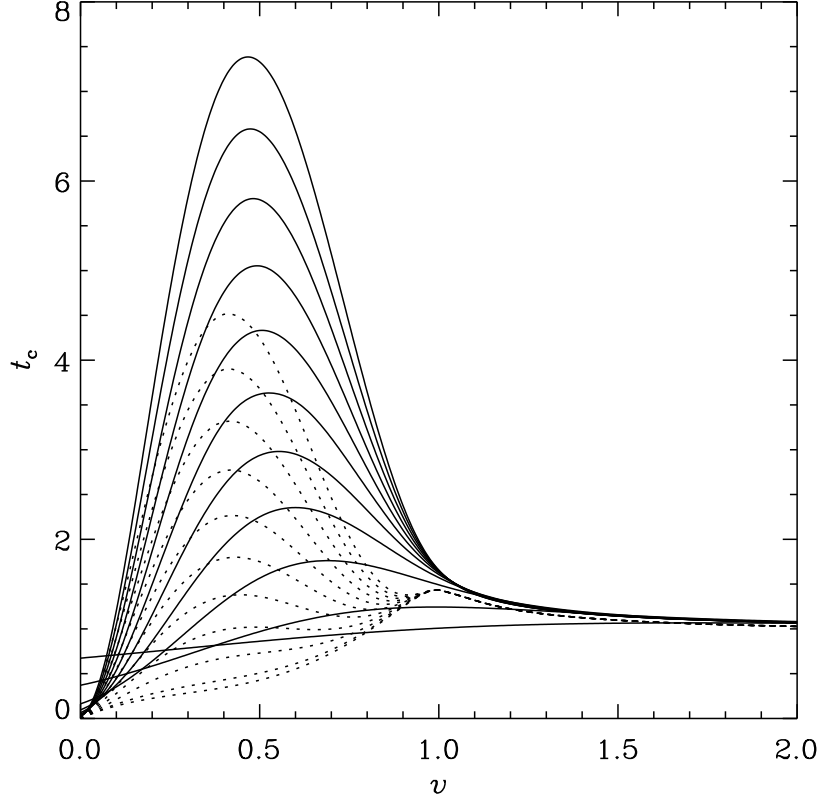


Figure 2. Steady dimensionless torque as a function of the dimensionless drift speed v for $p = 0.5, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ ($p = 0.5$ is the flattest curve). The figure is symmetrical about $v = 0$. The approximate relation (41) is shown as dotted lines for the same values of p .

$$\Phi' \approx -GM_p(r_c^2 s^2 + x^2)^{-1/2} + GM_p r_c^{-1} \cos \varphi, \quad (59)$$

where $s = 2 \sin(\varphi/2)$. The x -dependence in this expression is needed, of course, to localize the singularity of the potential at the location of the planet. In the case of the non-coorbital corotation resonance it was possible to treat the perturbing potential as being independent of x .

We construct an approximation to the flow in the coorbital region by taking the leading-order radial and azimuthal velocity perturbations to be

$$u' = -\frac{1}{2rB} \frac{\partial \Phi'}{\partial \varphi}, \quad v' = \frac{1}{2\Omega} \frac{\partial \Phi'}{\partial x}. \quad (60)$$

These expressions derive from the linearized fluid dynamical equations in the corotation region, when terms proportional to x and those associated with pressure and viscosity are neglected. The expression for u' is identical to equation (12), but the equivalent for v' is also needed for consistency near the planet. Thus the leading-order velocity field in the co-moving frame is

$$\mathbf{u} = \left\{ 2qr\Omega \sin \varphi \left[1 - \left(s^2 + \frac{x^2}{r^2} \right)^{-3/2} \right] - \frac{dr_c}{dt} \right\} \mathbf{e}_r + \left[\frac{1}{2} q\Omega x \left(s^2 + \frac{x^2}{r^2} \right)^{-3/2} - \frac{3}{2} \Omega x \right] \mathbf{e}_\varphi, \quad (61)$$

where r and Ω are evaluated at $r = r_c$, and we have specialized to the case of a Keplerian disc. Fig. 4 shows some streamlines, based on this velocity field, for an inwardly migrating planet. There is a trapped region on the leading side of the planet, and there are open streamlines that pass the planet on its trailing side. This behaviour is a consequence of the higher (lower) magnitude of radial velocity that results on the trailing (leading) side of the planet when the inward migration velocity is combined with the velocity (u', v') . A similar asymmetric behaviour was found by Artymowicz (2004).

In the absence of migration, the stagnation points can be identified as the standard Lagrange points:

$$\begin{aligned} \text{L1, L2:} \quad & x = \mp \left(\frac{q}{3} \right)^{1/3} r, \quad \varphi = 0 \\ \text{L3:} \quad & x = 0, \quad \varphi = \pi, \\ \text{L4, L5:} \quad & x = 0, \quad \varphi = \pm \frac{\pi}{3}. \end{aligned} \quad (62)$$

As the migration rate is increased from zero, a bifurcation occurs at $|dr_c/dt| \approx 1.45 q\Omega$, at which what were L3 and L4 (or L5, depending on the sign of dr_c/dt) merge and disappear. This behaviour can be seen in Figs 5 and 6, where the separatrices are plotted. However, a librating region continues to exist for any migration rate. This is an important difference with the case of the non-coorbital corotation resonance.

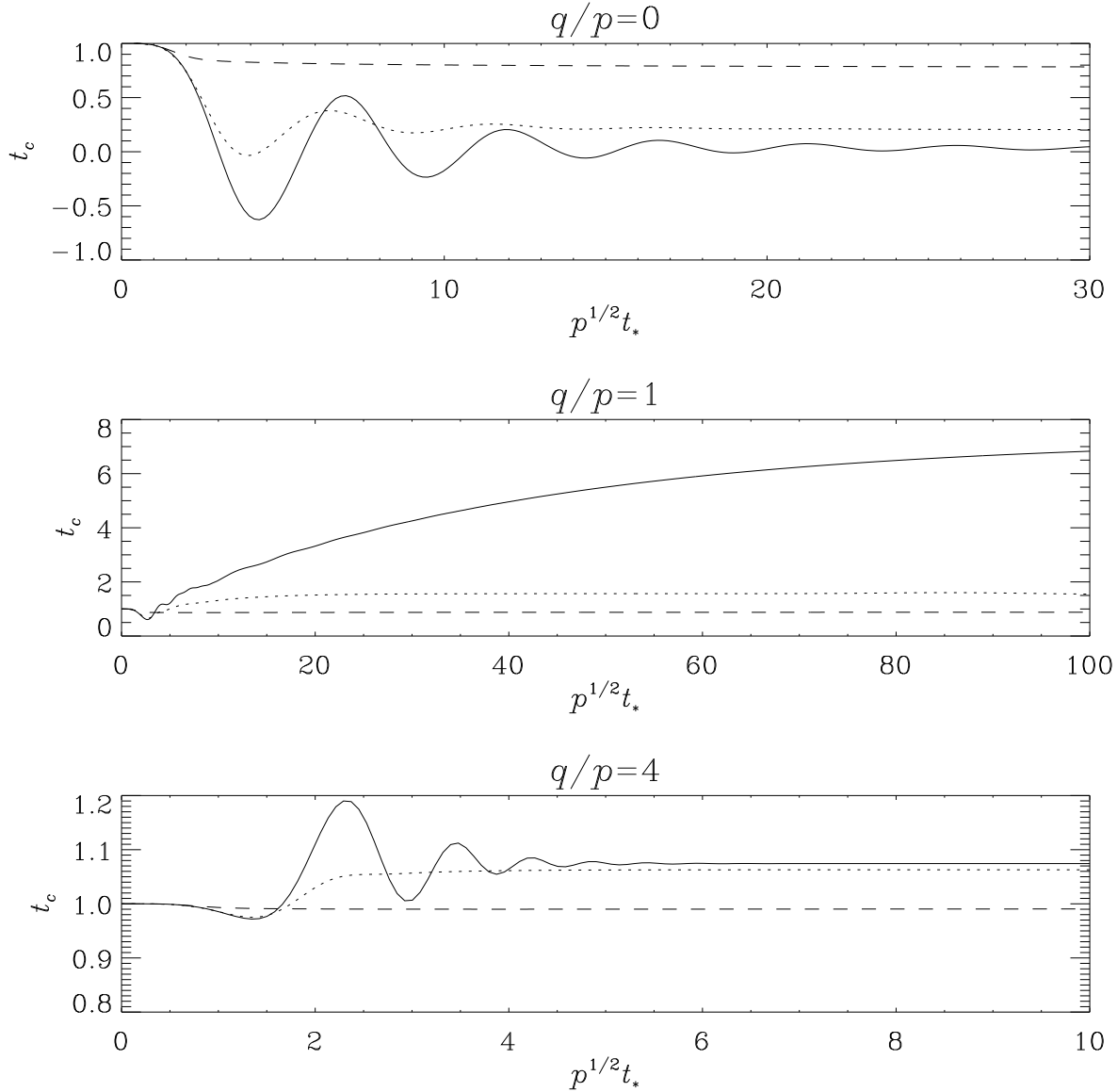


Figure 3. Time-dependence of the dimensionless torque for various parameter values. The scaling of the time variable is related to the characteristic libration time (in the case $v = 0$, at least). Upper panel: $v = 0$ and $p = 0.4$ (dashed), 2 (dotted) and 10 (solid). Middle panel: $v = 0.5$ and $p = 0.4$ (dashed), 2 (dotted) and 10 (solid). Lower panel: $v = 2$ and $p = 0.1$ (dashed), 0.5 (dotted) and 2.5 (solid).

9.2 Stability of migration with a radial velocity-dependent torque

If the coorbital torque depends on the migration rate \dot{a}_p , then \dot{a}_p satisfies a nonlinear equation of the form

$$\frac{dJ_p}{da_p} \dot{a}_p = T_c + T_L, \quad (63)$$

where $J_p = M_p(GM_*a_p)^{1/2}$ is the angular momentum of the planet in a circular orbit, and T_L is the ‘Lindblad’ torque that comes from the non-coorbital region of the disc and does not depend (or not sensitively, at least) on \dot{a}_p . If the migration is slow enough that the coorbital region is in a quasi-steady state, then T_c can be replaced with its steady value $T_{cs}(\dot{a}_p)$.

In Masset & Papaloizou (2003) it was argued that T_c is directly proportional to \dot{a}_p , but evaluated at a retarded time owing to the effects of libration. Our analysis suggests that the dependence on \dot{a}_p is nonlinear in general, although in the coorbital case it may not resemble Fig. 2 in detail because of the persistence of the librating region as \dot{a}_p is increased. Equation (63) may then have any number of solutions representing quasi-steady migration, and these solutions may be stable or unstable.

Suppose that the planet migrates quasi-steadily according to one of the solutions of equation (63) but is then perturbed slightly, for example by applying and removing an extraneous torque. The corotation torque will take some time to adjust to these changes. If it relaxes monotonically

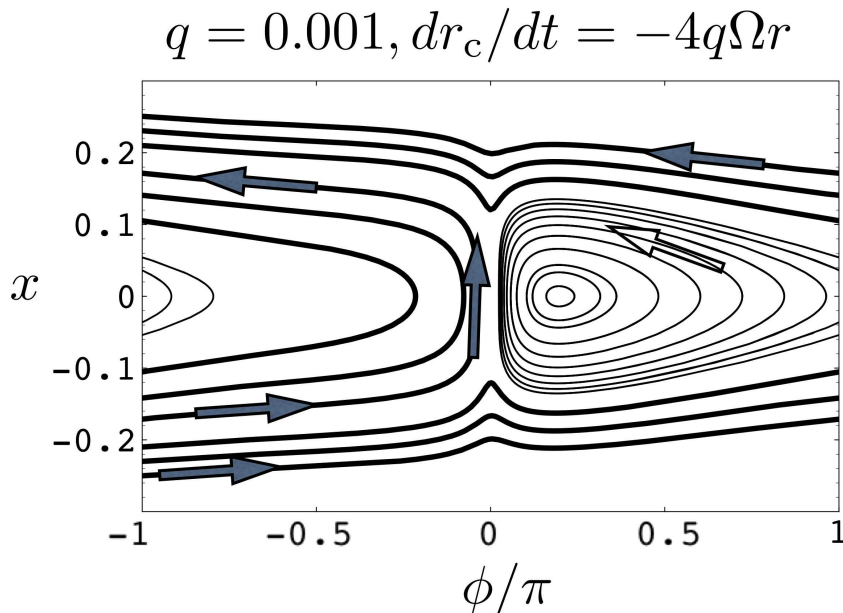


Figure 4. Some streamlines in the coorbital region for the case of a planet with mass ratio $q = 0.001$ and inward migration rate $dr_c/dt = -4q\Omega r$. The figure is based on the approximation (61) to the motion in the comoving frame with the planet located at the origin. The light streamlines are closed and trapped by the planet; the heavy streamlines are open. The trapped region is centred on the leading side of the planet and persists (but diminishes in size) at higher migration rates. The lowest streamline plotted is traced by the solid arrows, as it wraps in azimuth. This streamline circulates around the central star and then moves outwards, passing the planet, and continues outwards.

towards the steady value $T_{cs}(\dot{a}_p)$, then it can be shown that the migration is stable if

$$\frac{dT_{cs}}{d\dot{a}_p} < \frac{dJ_p}{da_p}, \quad (64)$$

and unstable in the opposite case. [Formally this can be shown by combining equation (63) with a relaxation equation such as $dT_c/dt = (T_{cs} - T_c)/\tau$.] If the quasi-steady solutions of equation (63) are viewed as the points of intersection of the graph $T_{cs}(\dot{a}_p)$ with a straight line of gradient dJ_p/da , then the solutions alternate in stability as the curve cuts the line alternately from below and from above. While in the analysis of Masset & Papaloizou (2003) there is only one solution and it is either stable or unstable, here the possibility exists, in principle, of the planet finding an alternative migration rate that is stable.

10 SUMMARY AND DISCUSSION

We have investigated the effects of the radial migration of a planet through a disc on the corotation resonance and the associated torque. Our analysis deals mainly with non-

coorbital corotation resonances, which are critical to eccentricity evolution (Goldreich & Tremaine 1980). It was carried out using semi-analytical methods that account for the effects of gas pressure, turbulent viscosity and nonlinear saturation, as well the migration of the planet. Our approach does not account for the possible effects of shocks that cause changes in vortensity, which can be studied instead through nonlinear simulations.

In the absence of radial motion, the corotational flow pattern consists of islands of closed streamlines, surrounded by a modified Keplerian flow that circulates around the star. The radial motion of the planet causes the resonance to drift through the disc and modifies the streamlines in the comoving frame, separating the closed regions and allowing radial flow through the coorbital region (see Fig. 1). At faster migration rates the closed regions are destroyed. The characteristic radial velocity for this transition is the radial width of the libration region divided by the libration time-scale.

We find that the ratio of the resonant torque in a steady state to the value given by GT79 depends essentially on two dimensionless parameters. One of these (p) is proportional to the ratio of the characteristic time-scale of viscous diffusion across the librating region of the resonance to the charac-

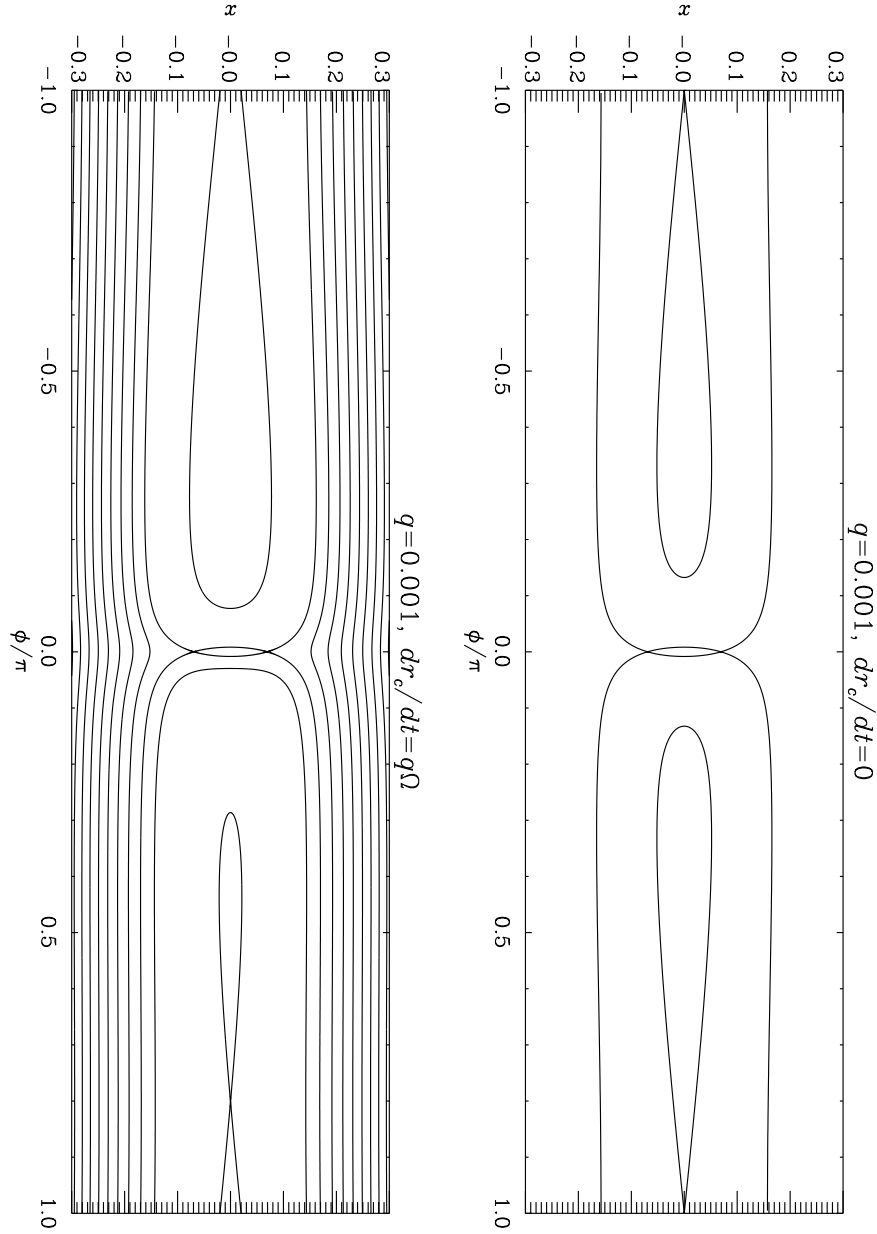


Figure 5. Separatrices of streamlines in the coorbital region. The figure refers to the approximation (61) to the motion in the comoving frame, for various values of the migration rate. The streamlines continuously fill the regions between separatrices.

teristic time-scale of libration (raised to the power $2/3$); in the absence of migration, this parameter alone determines the degree of saturation of the resonance. The second parameter (v) is proportional to the ratio of the drift speed of the resonance to the characteristic radial velocity in the librating region; this determines the shape of the streamlines. When the drift speed is comparable to the libration speed and the viscosity is small, the torque achieved in a steady state can be much larger than the unsaturated value in the absence of migration, but is still proportional to the large-scale vortensity gradient in the disc.

The resulting corotation torque is a nonlinear function of drift speed, which peaks at the characteristic velocity (see Fig. 2). The radial motion of a planet generally enhances the corotation torque by an amount that varies inversely with the turbulent viscosity, in the limit of small viscosity. This torque does not change sign under reversal of the direction of radial motion. The corotation torque is proportional to the vortensity gradient, and the dependence on viscosity can be understood in terms of the vortensity redistribution in the corotation region. The trapped material attempts to preserve its original vortensity as it migrates radially into lo-

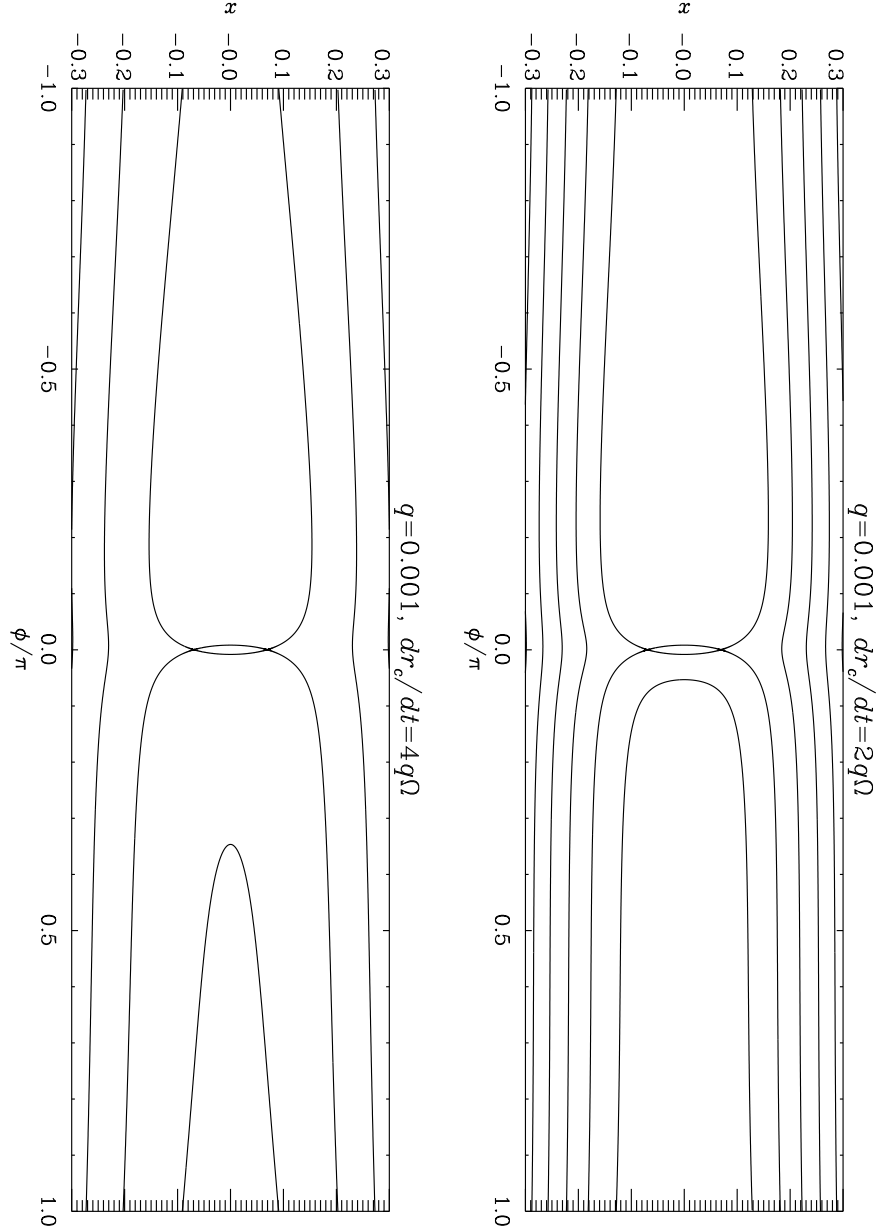


Figure 6. Continuation of Fig. 5.

cations of different vortensity. As the contrast in vortensity between the trapped regions and surrounding background increases, the torque increases as well, but is limited by viscous diffusion between the two regions. If the viscosity is very small this steady state may take so long to be achieved that the torque needs to be considered in a time-dependent sense. A generally smaller contribution to the torque arises from material that passes across the coorbital region, between the islands of closed streamlines.

We suggest that the physics of the non-coorbital resonance can provide insight into the more complicated situation for the coorbital region. There are similarities between

the flow patterns in the two cases (Figs 1 and 4). In each case, there are regions of closed streamlines that move with the planet, as well as open streamlines that pass through the corotation region. We expect that the torque that results from the closed streamlines in the coorbital case also involves the contrast between the background vortensity and the advected vortensity. The torque from the closed streamlines would then be again inversely proportional to viscosity, in the small viscosity limit, provided that a large vortensity anomaly is able to accumulate. There are certainly differences between the two cases. In the coorbital region, many different azimuthal components of the potential participate

and there is a singularity present due to the presence of the planet. Because of the singularity, there is always a region of trapped streamlines on one side of the planet for any migration rate and this region will contribute to the coorbital torque.

The analogy between the non-coorbital and coorbital cases also suggests that there may be, in some circumstances, several possible migration rates in a given system because the corotation torque is a nonlinear function of the migration rate. Which one is achieved depends on the initial conditions and on considerations of stability, as discussed in Section 9.2. It is possible that fast migration does occur under some appropriate conditions. Masset & Papaloizou (2003) emphasized the importance of a mass deficit in a partially cleared gap. Our analysis does not involve a gap, but considers the role of advected vortensity. The mass deficit would certainly contribute to the advected vortensity contrast with the background, but may not be necessary. More accurate modelling of the coorbital region is required to determine its contribution to the migration rate.

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APPENDIX A: ASYMPTOTIC SOLUTION OF THE FORCED ADVECTION–DIFFUSION EQUATION

We consider the advection–diffusion equation

$$\frac{\partial Q'}{\partial t} + \mathbf{u} \cdot \nabla Q' - \nu \nabla^2 Q' = S, \quad (\text{A1})$$

where \mathbf{u} is a specified steady, two-dimensional velocity field satisfying $\nabla \cdot \mathbf{u} = 0$, $\nu > 0$ is a uniform diffusivity, and S is a specified steady source function. A streamline of \mathbf{u} that does not include a stagnation point can either close on itself or be open to infinity. We assume that \mathbf{u} has a region \mathcal{C} of closed streamlines bounded by a separatrix and surrounded by a region \mathcal{O} of open streamlines; the argument is easily generalized to allow for multiple closed regions. The boundary condition on open streamlines is that $Q' \rightarrow 0$ far upstream. The interaction between the source and the response should be effectively confined to a limited region of space, either because S decays at large distance or because rapid phase variations lead to cancellation.

Let χ be a streamfunction such that $\mathbf{u} = \nabla \chi \times \mathbf{e}_z$, where \mathbf{e}_z is the unit vector normal to the plane of the flow, and let λ be the travel time along a streamline (measured from its intersection with an arbitrary continuous curve). Then (λ, χ) are coordinates that cover \mathcal{C} and \mathcal{O} separately. In region \mathcal{C} , $\lambda \in [0, \Lambda(\chi))$ is a periodic variable on each streamline; in region \mathcal{O} , $\lambda \in (-\infty, \infty)$. The element of area is $dA = d\lambda d\chi$.

A steady solution of equation (A1) satisfies

$$\frac{\partial Q'}{\partial \lambda} - \nu \nabla^2 Q' = S. \quad (\text{A2})$$

We seek an asymptotic solution for large Reynolds number (small ν). A naïve expansion of the form

$$Q' = Q'_0(\lambda, \chi) + \nu Q'_1(\lambda, \chi) + \dots \quad (\text{A3})$$

would imply, at $O(\nu^0)$,

$$\frac{\partial Q'_0}{\partial \lambda} = S. \quad (\text{A4})$$

This generally fails in region \mathcal{C} because it requires the source function to satisfy the solvability condition

$$0 = \int_0^{\Lambda(\chi)} S d\lambda \quad (\text{A5})$$

on each streamline in \mathcal{C} . Instead, the solution is of the form

$$Q' = \nu^{-1} Q'_{-1}(\chi) + Q'_0(\lambda, \chi) + \nu Q'_1(\lambda, \chi) + \dots \quad (\text{A6})$$

Equation (A2) at $O(\nu^0)$ and $O(\nu^1)$ then gives

$$\frac{\partial Q'_0}{\partial \lambda} - \nabla^2 Q'_{-1} = S, \quad (\text{A7})$$

$$\frac{\partial Q'_1}{\partial \lambda} - \nabla^2 Q'_0 = 0. \quad (\text{A8})$$

In region \mathcal{O} the upstream boundary condition $Q' \rightarrow 0$ as $\lambda \rightarrow -\infty$ requires that Q'_{-1} vanish identically. The ‘anomalous’ response Q'_{-1} is confined to the region of closed streamlines on which there is a net forcing and the response builds up until limited by viscous diffusion.

In region \mathcal{O} we therefore have

$$Q'_0(\lambda, \chi) = \int_{-\infty}^{\lambda} S(\lambda', \chi) d\lambda'. \quad (\text{A9})$$

In region \mathcal{C} the solvability conditions

$$-\int_0^{\Lambda(\chi)} \nabla^2 Q'_{-1} d\lambda = \int_0^{\Lambda(\chi)} S d\lambda, \quad (\text{A10})$$

$$-\int_0^{\Lambda(\chi)} \nabla^2 Q'_0 d\lambda = 0 \quad (\text{A11})$$

apply on each streamline. Let $A(\chi)$ denote the area enclosed by the (closed) streamline $C(\chi)$ on which $\chi = \text{constant}$. Then we have (after integration with respect to χ)

$$-\int_{A(\chi)} \nabla^2 Q'_{-1} dA = \int_{A(\chi)} S dA, \quad (\text{A12})$$

The divergence theorem implies

$$\begin{aligned} \int_{A(\chi)} \nabla^2 Q'_{-1} dA &= \int_{C(\chi)} \nabla Q'_{-1} \cdot \mathbf{n} ds \\ &= \frac{dQ'_{-1}}{d\chi} \int_{C(\chi)} \nabla \chi \cdot \mathbf{n} ds \\ &= \frac{dQ'_{-1}}{d\chi} \int_{A(\chi)} \nabla^2 \chi dA, \end{aligned} \quad (\text{A13})$$

and so

$$\frac{dQ'_{-1}}{d\chi} = - \int_{A(\chi)} S dA \Big/ \int_{A(\chi)} \nabla^2 \chi dA. \quad (\text{A14})$$

For continuity with region \mathcal{O} in which $Q'_{-1} = 0$, we have $Q'_{-1}(\chi_s) = 0$, where $\chi = \chi_s$ is the separatrix. Therefore we have uniquely determined Q'_{-1} in \mathcal{C} and Q'_0 in \mathcal{O} . It should be noted, however, that Q'_0 as given by equation (A9) will be viscously attenuated at sufficiently large λ .

Now consider the integral

$$I = \int_{\mathcal{R}} Q' S dA \quad (\text{A15})$$

over a region \mathcal{R} containing the part of the flow (including the whole of \mathcal{C}) where the interaction is significant. This integral is directly related to the corotation torque. In a steady state

$$\begin{aligned} I &= \int_{\mathcal{R}} Q' (\mathbf{u} \cdot \nabla Q' - \nu \nabla^2 Q') dA \\ &= \int_{\partial \mathcal{R}} \left(\frac{1}{2} Q'^2 \mathbf{u} - \nu Q' \nabla Q' \right) \cdot \mathbf{n} ds + \int_{\mathcal{R}} \nu |\nabla Q'|^2 dA. \end{aligned} \quad (\text{A16})$$

In terms of the asymptotic solution we find

$$I = \nu^{-1} I_{-1} + I_0 + \dots, \quad (\text{A17})$$

with

$$I_{-1} = \int_{\mathcal{C}} |\nabla Q'_{-1}|^2 dA, \quad (\text{A18})$$

$$I_0 = \int_{\partial \mathcal{R}} \frac{1}{2} Q_0'^2 \mathbf{u} \cdot \mathbf{n} ds + 2 \int_{\mathcal{C}} (\nabla Q'_{-1}) \cdot \nabla Q'_0 dA. \quad (\text{A19})$$

Although the advective flux in the first term in the expression for I_0 is converted into a viscous flux as the disturbance is attenuated, this term can be evaluated as if viscosity had no effect and the disturbance was advected to $\lambda \rightarrow +\infty$. The second term vanishes because

$$\int_{\mathcal{C}} (\nabla Q'_{-1}) \cdot \nabla Q'_0 dA = - \int_{\mathcal{C}} Q'_{-1} \nabla^2 Q'_0 dA$$

$$\begin{aligned} &= - \int Q'_{-1} \int \nabla^2 Q'_0 d\lambda d\chi \\ &= 0, \end{aligned} \quad (\text{A20})$$

where we use the boundary condition $Q'_{-1} = 0$ on the separatrix and the solvability condition (A11).

Thus the leading contribution

$$I_{-1} = \int_{\mathcal{C}} \left(\frac{dQ'_{-1}}{d\chi} \right)^2 |\nabla \chi|^2 dA \quad (\text{A21})$$

comes entirely from the closed region, while the next term

$$I_0 = \int_{\mathcal{O}} \frac{1}{2} \left[\int_{-\infty}^{\infty} S d\lambda \right]^2 d\chi \quad (\text{A22})$$

comes entirely from the open region.